Toeplitz determinants, multicritical random partitions and the dPII hierarchy

S. Tarricone , based on a collaboration with T. Chouteau (Unviersité d'Angers) ArXiv:2211.16898

Institut de Physique Théorique, CEA-CNRS, Université Paris-Saclay

Toeplitz determinants in random partitions

For any $N \geq 1$, we consider the symbol $\varphi^{(N)}[\theta_1, \dots, \theta_N](z) = \varphi(z) = e^{w(z)}$, with

$$w(z) = v(z) + v(z^{-1}) \quad \text{and} \quad v(z) = \sum_{j=1}^{N} \frac{\theta_j}{j} z^j, \ \theta_j \in \mathbb{R}, \ z \in S^1.$$

The Toeplitz determinants associated to the symbol $\varphi(z)$ are defined as $D_n = \det(T_n(\varphi))$ with $T_n(\varphi)$ being the n-th Toeplitz matrix

$$T_n(\varphi)_{i,j} = \varphi_{i-j}, \quad i,j = 0, \dots, n$$

where for every $k \in \mathbb{Z}$, φ_k is the k-th Fourier coefficient of $\varphi(z)$, namely

$$\varphi_k = \int_{-\pi}^{\pi} e^{-ik\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi}, \text{ so that } \sum_{k \in \mathbb{Z}} \varphi_k z^k = \varphi(z).$$

[7] Consider now on the set of all integer partitions the Schur measures

$$\mathbb{P}_{\text{Sc.}}(\{\lambda\}) = Z^{-1} s_\lambda [\theta_1, \dots, \theta_N]^2,$$

where s_λ is the Schur function indexed by λ which can be computed as $s_\lambda[\theta_1, \dots, \theta_N] = \det_{i,j} h_{\lambda_i - i + j}[\theta_1, \dots, \theta_N]$, with $\sum_{k \geq 0} h_k z^k = e^{v(z)}$, and $Z = e^{\sum_{i=1}^N \frac{\theta_i^2}{i}}$.

Remark For $N = 1$, with $\theta_1 = \theta$, we have $\mathbb{P}_{\text{Sc.}}(\{\lambda\}) = \mathbb{P}_{\text{P.Pl.}}(\{\lambda\}) = e^{-\theta^2} \left(\frac{\theta^{|\lambda|} F_\lambda}{|\lambda|!} \right)^2$, where $|\lambda|$ is the size of λ and F_λ is the number of standard Young tableau of shape λ.

[4] In this setting, denoting by $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ a generic integer partition and by $\lambda' = (\lambda_1' \geq \lambda_2' \geq \dots \geq 0)$ its conjugate partition (namely such that $\lambda_j' = |i : \lambda_i \geq j|$), we have the following characterization

$$q_n = \mathbb{P}_{\text{Sc.}}(\lambda_1' \leq n) = e^{-\sum_{j=1}^N \theta_j^2/j} D_{n-1} \quad \text{and} \quad r_n = \mathbb{P}_{\text{Sc.}}(\lambda_1 \leq n) = e^{-\sum_{j=1}^N \tilde{\theta}_j^2/j} \widetilde{D}_{n-1},$$

where the Toeplitz determinant $\widetilde{D}_n$ is instead associated to the symbol $\widetilde{\varphi}(z) = e^{\tilde{w}(z)}$ built up by taking $\widetilde{\theta}_i = (-1)^{i-1}\theta_i$ and $\tilde{w}(z) = \tilde{v}(z) + \tilde{v}(z^{-1})$ where $\tilde{v}(z)$ is nothing than $v(z)$ with θ_i replaced by $\tilde{\theta}_i$.

Multicritical limiting behavior

[4] Let $\theta_i = (-1)^{i+1} \frac{(N-1)!(N+1)!}{(N-i)!(N+i)!} \theta$. Then the limiting behavior of r_n is described by

$$\lim_{\theta \to +\infty} \mathbb{P}_{\text{Sc.}} \left(\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2N+1}}} < t \right) = F_N(t) = \det(1 - \mathscr{K}_{\text{Ai}_N}|_{(t,\infty)}), \quad b = \frac{N+1}{N}, d = \binom{2N}{N-1}^{-1}.$$

[6] The Fredholm determinant $F_N(t)$ satisfies

$$\partial_t^2 \log F_N(t) = -u^2((-1)^{N+1}(t)),$$

$u(t)$ solving the N-th member of the Painlevé II hierarchy with $u(t) \underset{t \to +\infty}{\sim} \text{Ai}_N(t)$.

Orthogonal polynomials on S^1

We consider the measure for $z = e^{i\alpha} \in S^1$ given by

$$d\mu(\alpha) = \varphi(e^{i\alpha}) \frac{d\alpha}{2\pi} = e^{w(e^{i\alpha})} \frac{d\alpha}{2\pi}.$$

The family $\{p_n(z)\}_{n \in \mathbb{N}}$ of orthogonal polynomials on the unitary circle w.r.t. the measure $d\mu(\alpha)$ is a sequence of polynomials of increasing degree

$$p_n(z) = \kappa_n z^n + \dots \kappa_0, \quad \kappa_n > 0$$

such that the following relation holds for any index k, h

$$\int_{-\pi}^{\pi} \overline{p_k(e^{i\alpha})} p_h(e^{i\alpha}) \frac{d\mu(\alpha)}{2\pi} = \delta_{k,h}.$$

The analogue monic orthogonal polynomials $\pi_n(z)$ are $p_n(z) = \kappa_n \pi_n(z)$.
For every $n \geq 1$ the following formula holds

$$p_n(z) = \frac{1}{\sqrt{D_n D_{n-1}}} \begin{vmatrix} \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n+1} & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \dots & \varphi_{-n+2} & \varphi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \dots & \varphi_0 & \varphi_{-1} \\ 1 & z & \dots & z^{n-1} & z^n \end{vmatrix} \implies \frac{D_{n-1}}{D_n} = \kappa_n^2.$$

Riemann–Hilbert problem For any fixed $n \geq 1$, the function $Y(z) = Y(z, n; \theta_i) : \mathbb{C} \to GL(2, \mathbb{C})$ has the following properties

(1) $Y(z)$ is analytic for every $z \in \mathbb{C} \setminus S^1$;

(2) $Y(z)$ has continuous boundary values $Y_\pm(z)$ are related for all $z \in S^1$ through

$$Y_+(z) = Y_-(z) J_Y(z), \quad \text{with} \quad J_Y(z) = \begin{pmatrix} 1 & z^{-n} e^{w(z)} \\ 0 & 1 \end{pmatrix};$$

(3) $Y(z)$ is normalized as $Y(z) \sim \left(I + \sum_{j=1}^\infty \frac{Y_j(n,\theta_i)}{z^j} \right) z^{n\sigma_3}, \quad z \to \infty,$

where σ_3 denotes the Pauli's matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

[3] The Riemann–Hilbert problem admits a unique solution $Y(z)$ written as

$$Y(z) = \begin{pmatrix} \pi_n(z) & \mathscr{C}\left(y^{-n} \pi_n(y) e^{w(y)}\right)(z) \\ -\kappa_{n-1}^2 \pi_{n-1}^*(z) & -\kappa_{n-1}^2 \mathscr{C}\left(y^{-n} \pi_{n-1}^*(y) e^{w(y)}\right)(z) \end{pmatrix},$$

where $\pi_{n-1}^*(z)$ is defined as the polynomial of the same degree of $\pi_{n-1}(z)$ such that $\pi_{n-1}^*(z) = z^n \overline{\pi_{n-1}(\bar{z}^{-1})}$. and $(\mathscr{C} f(y))(z)$ is the Cauchy transform of f

$$(\mathscr{C} f(y))(z) = \frac{1}{2\pi i} \int_{S^1} \frac{f(y)}{y - z} dy.$$

Moreover, $\det(Y(z)) \equiv 1$.

In particular, from the fact that $\det(Y(0, n; \theta_i)) = 1$ we obtain for every $n \geq 1$

$$\frac{D_{n-2} D_n}{D_{n-1}^2} = 1 - x_n^2, \text{ with } x_n = \pi_n(0) \in \mathbb{R}.$$

The connection with the discrete Painlevé II hierarchy

We construct the solution of another Riemann–Hilbert problem, which has in particular z, n-independent jump matrix

$$\Psi(z, n; \theta_i) = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_n^{-2} \end{pmatrix} Y(z, n; \theta_i) \begin{pmatrix} 1 & 0 \\ 0 & z^n \end{pmatrix} e^{w(z) \frac{\sigma_3}{2}}.$$

Thanks to that, the function $\Psi(z, n; \theta_i) = \Psi(z, n)$ is proved to solve the linear system

$$\Psi(z, n+1) = U(z, n) \Psi(z, n), \quad \partial_z \Psi(z, n) = T(z, n) \Psi(z, n)$$

with

$$U(z, n) = \begin{pmatrix} z + x_n x_{n+1} & -x_{n+1} \\ -(1 - x_{n+1}^2) x_n & 1 - x_{n+1}^2 \end{pmatrix} = \sigma_+ z + U_0(n), \quad T(z, n) = \sum_{k=1}^{2N+1} T_k(n) z^{N-k},$$

where $T_1(n) = \frac{\theta_N}{2} \sigma_3$ and $T_j(n) = -K(n) T_{2N+2-j}(n) K(n)^{-1}$, for $j = 1, \dots, N$, $T_{N+1}(n) = -K(n) T_{N+1}(n) K(n)^{-1} + n I_2$. This systems turns out to be a Lax pair for the discrete Painlevé II hierarchy. Indeed, the compatibility condition

$$\sigma_+ = T(n+1, z) U(n, z) - U(n, z) T(n, z)$$

- gives a system of discrete (in n) equations for $T_k^{ij}(n), i, j \in \{1, 2\}$ for $k = 1, \dots, N+1$ that determines all of them in terms of $x_{n \pm j}, j = -N, \dots, N$ recursively (on k), starting from the initial condition for $T_1(n)$.
- Plugging the form obtained for the last coefficient $T_{N+1}(n)$ into the equation for $T_{N+1}(n)$ given above, it finally gives a nonlinear $2N$ order discrete equation for x_n which is the N-th equation of the discrete Painlevé II hierarchy.

For any fixed $N \geq 1$, for the Toeplitz determinants $D_n, n \geq 1$, we have the following recursion relation

$$\frac{D_n D_{n-2}}{D_{n-1}^2} = 1 - x_n^2$$

where x_n solves the $2N$ order nonlinear discrete equation

$$n x_n + \left(v_n + v_n \text{Perm}_n - 2 x_n \Delta^{-1} (x_n - (\Delta + I) x_n \text{Perm}_n) \right) L^N(0) = 0$$

where L is a discrete recursion operator that acts as follows

$$L(u_n) = \left(x_{n+1} \left(2\Delta^{-1} + I \right) \left((\Delta + I) x_n \text{Perm}_n - x_n \right) + v_{n+1} (\Delta + I) - x_n x_{n+1} \right) u,$$

and $L(0) = \theta_N x_{n+1}$. Here $v_n = 1 - x_n^2$, Δ denotes the difference operator $\Delta : u_n \to u_{n+1} - u_n$ and Perm_n is the transformation

$$\begin{aligned} \text{Perm}_n : \quad \mathbb{C}\left[(x_j)_{j \in [[0,2n]]} \right] &\longrightarrow \mathbb{C}\left[(x_j)_{j \in [[0,2n]]} \right] \\ P\left((x_{n+j})_{-n \leqslant j \leqslant n} \right) &\longmapsto P\left((x_{n-j})_{-n \leqslant j \leqslant n} \right). \end{aligned}$$

$\boxed{N=1}$ In this case x_n solves the second order discrete equation

$$\text{dPII}_1 \quad \theta_1 (x_n^2 - 1)(x_{n+1} + x_{n-1}) = n x_n.$$

By defining $x_n = (-1)^n \theta^{-1/3} u(t)$ with $t = (n - 2\theta)\theta^{-1/3}$ and taking the limit $\theta \to +\infty$, dPII_1 scales to the Painlevé II equation $u''(t) = 2u^3(t) + t u(t)$ and the recursion for the Toeplitz determinants to $\partial_t^2 \log F_1(t) = -u^2(t)$.

$\boxed{N=2}$ In this case x_n solves the fourth order discrete equation

$$\text{dPII}_2 \quad n x_n + \theta_1 v_n (x_{n+1} + x_{n-1}) + \theta_2 v_n \left(x_{n+2} v_{n+1} + x_{n-2} v_{n-1} - x_n (x_{n+1} + x_{n-1})^2 \right) = 0.$$

By scaling $\theta_1 = \theta$, $\theta_2 = \frac{\theta}{4}$, defining $x_n = (-1)^n \left(\frac{\theta}{4}\right)^{-1/5} u(t)$ with $t = \left(n - \frac{3}{2}\theta\right) \theta^{-\frac{1}{5}} 4^{\frac{1}{5}}$ and taking the limit $\theta \to +\infty$, dPII_2 scales to the second equation of the Painlevé II hierarchy $u'''' - 10u(u')^2 - 10u^2 u'' + 6u^5 = -tu$ and the recursion for the Toeplitz determinants to $\partial_t^2 \log F_2(t) = -u^2(t)$.

Remark For $N = 1$ the result was already proved from different authors in different ways [1, 2, 5, 8].

References

[1] Adler, Moerbeke. "Recursion Relations for Unitary Integrals, Combinatorics and the Toeplitz Lattice". *Communications in Mathematical Physics* 237 (2003), no. 3.

[2] Baik. "Riemann-Hilbert problems for last passage percolation". *Contemp. Math.* 326 (2001), no. 8.

[3] Baik, Deift & Johansson. "On the distribution of the length of the longest increasing subsequence of random permutations". *J. of the Am. Math. Soc.* 12 (1999), no. 4.

[4] Betea, Bouttier & Walsh. "Multicritical random partitions". *33st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2021)* 85B (2022).

[5] Borodin. "Discrete gap probabilities and discrete Painlevé equations". *Duke Math. J.* 113 (2003), no. 3.

[6] Cafasso, Claeys & Girotti. "Fredholm Determinant Solutions of the Painlevé II Hierarchy and Gap Probabilities of Determinantal Point Processes". *Int. Math. Res. Not.* IMRN 168 (2019).

[7] Okounkov. "Infinite wedge and random partitions". *Selecta Math.* (N.S.) 7.1 (2001).

[8] Periwal & Shevitz. "Exactly solvable unitary matrix models: Multicritical potentials and correlations". *Nuclear Physics B* 334 (1990), no. 3.